

# Applying a rigorous quasi-steady state approximation method for proving the absence of oscillations in models of genetic circuits<sup>\*</sup>

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**Abstract.** In this paper, a rigorous quasi-steady state approximation method is applied in order to study the absence of oscillations in a family of models describing a gene regulated by a polymer of its own protein. A result presented in a former paper at *Algebraic Biology 2007* is thereby generalized. The rigorous method is illustrated over the basic enzymatic reaction.

## 1 Introduction

In a former paper [1], we studied a simple family of models depending on an integer parameter  $n$  and featuring a negative feedback loop, one of the core ingredients for generating oscillations [2]. These abstract models are closely related to models studied by Goodwin and Griffith in the 60's [3–5]. Griffith considered a model of a gene regulated by a polymer formed of  $n$  copies of its own protein. We studied the same problem, but in a slightly more general case, where gene activation is not assumed to be fast. We eventually concluded with the absence of Poincaré-Andronov-Hopf bifurcation in our family of models for  $n \leq 8$  and their existence for  $n \geq 9$ . The absence/presence of Poincaré-Andronov-Hopf bifurcation for  $n \leq 8$  is a strong indicator for the absence/presence of oscillating trajectories. Extensive numerical experiments [6, 7] confirmed the absence of oscillations for  $n \leq 8$  and their existence for  $n \geq 9$ .

In this paper, the models are designed by means of systems of parametric nonlinear ordinary differential equations (ODE) [8]. The approach applied in [1] consisted in two steps: first simplifying the initial system of  $n + 2$  parametric ordinary differential equations as a reduced system of three ODE by means of a quasi-steady state approximation; second, studying the reduced model.

The idea of quasi-steady state approximation is simple: study the dynamics of the slow reactions, assuming that the fast ones are at quasi-equilibrium,

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thereby removing from the ODE system, the differential equations which describe the evolution of the variables at quasi-equilibrium. Many authors [9–12] state that carrying out rigorously this approximation is far from straightforward. We would rather say that there are different ways to perform this approximation and that it is the problem of ascertaining the domain of validity of each kind of approximation which is not straightforward.

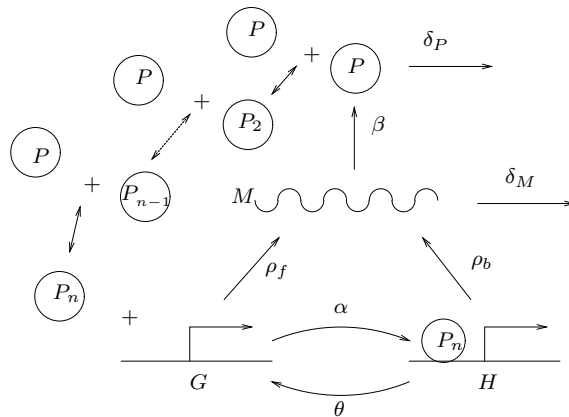
In another paper [13], the authors reformulated the methods of [9–12], which are equivalent, and made them fully algorithmic, by means of differential elimination methods [14–18]. An efficient implementation, based on [19], was developed by the third author.

In this paper, we show that the reduction method of [13] can be applied to our family of models. It yields a reduced model which contains that of [1] as a particular case: our new approximation is more precise. By a very concise proof, we show that the results obtained in [1] also hold for the new model. This paper gives us also the opportunity to widen the audience of our [13, DIFFREDUCE] algorithm by recalling its principle.

## 2 Our family of models

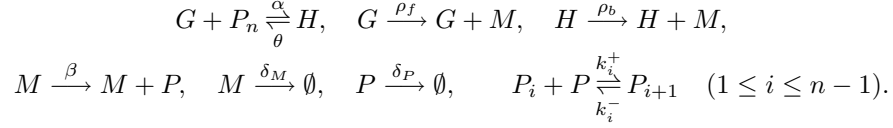
### 2.1 The initial model

A schematic model describing a single gene regulated by an order  $n$  polymer of its own protein is provided in Figure 1. It is borrowed from [1, page 68]. The variables  $G$  and  $H$  represent the state of the gene. The mRNA concentration and the concentration of the protein translated from the mRNA are represented by  $M$  and  $P$ . The  $n$  types of polymers of  $P$  are denoted by  $P = P_1, P_2, \dots, P_n$ . Greek letters represent parameters.



**Fig. 1.** A single gene regulated by a polymer of its protein

First, one translates the diagram as a system of generalized [20] chemical reactions (transcription and translation are not balanced reactions), introducing  $2(n-1)$  extra parameters  $k_i^-$ ,  $k_i^+$  ( $1 \leq i \leq n-1$ ):



The initial model is obtained by translating the above chemical reactions system as the following system of parametric ordinary differential equations, denoting  $A_i = (k_i^- P_{i+1} - k_i^+ P_i P)$ . Variables  $G$ ,  $H$ ,  $M$ ,  $P = P_1, \dots, P_n$  are time varying functions. The dot appearing over the variables, in the equations left hand sides, denotes the derivative w.r.t. the time.

$$\begin{aligned} \dot{G} &= \theta H - \alpha G P_n, \\ \dot{H} &= -\theta H + \alpha G P_n, \\ \dot{M} &= \rho_f G + \rho_b H - \delta_M M, \\ \dot{P} &= \beta M - \delta_P P + 2A_1 + A_2 + \dots + A_{n-1}, \\ \dot{P}_i &= -A_{i-1} + A_i \quad (2 \leq i \leq n-1), \\ \dot{P}_n &= -A_{n-1} + \theta H - \alpha G P_n \end{aligned} \tag{1}$$

The variables  $G$  and  $H$  should be viewed as “random variables” instead of concentrations. They are bound by the relation  $G + H = \gamma_0$  where  $\gamma_0$  is a constant equal to the total quantity of the gene. In [1], the variable  $H$  was replaced by  $\gamma_0 - G$ .

## 2.2 The new quasi-steady state approximation

The quasi-steady state approximation that is performed in this paper relies on the following hypotheses: the  $n-1$  chemical reactions describing the polymerization of the protein are fast while the other ones are slow. This happens when the parameters  $k_i^+$ ,  $k_i^-$  are much larger than the other parameters.

These hypotheses permit us to compute a reduced model by the following method. First one replaces, in the initial model, the contributions of the fast reactions by new variables  $F_i$  ( $1 \leq i \leq n-1$ ). This just amounts to rewriting  $A_i = F_i$  in system (1). Then one adds the following algebraic equations to the system, in order to express the pre-equilibrium conditions:

$$0 = k_i^+ P P_i - k_i^- P_{i+1}, \quad (1 \leq i \leq n-1)$$

Then one eliminates the new variables  $F_i$  from the above differential-algebraic system, thanks<sup>3</sup> to [13, DIFFREDUCE]. One is led to the raw reduced model:

$$\begin{aligned}\dot{G} &= \theta H - \alpha K_{n-1} P^n G, \\ \dot{H} &= -\theta H + \alpha K_{n-1} P^n G, \\ \dot{M} &= \rho_b H + \rho_f G - \delta_M M, \\ \dot{P} &= \frac{n \theta H - n \alpha K_{n-1} P^n G - \delta_P P + \beta M}{\sum_{i=0}^{n-1} (i+1)^2 K_i P^i}\end{aligned}\tag{2}$$

where  $K_i = \frac{k_1^+ \dots k_i^+}{k_1^- \dots k_i^-}$  with the convention  $K_0 = 1$ .

### 2.3 Parameters reduction

The raw reduced model (2) can now be simplified by rescaling all parameters and variables. The following equations express the old variables as functions of the new ones, which are overlined:

$$\begin{aligned}\theta &= \overline{\theta} \overline{\delta_M}, \quad \beta = \overline{\beta} \overline{\delta_M}, \quad \delta_P = \overline{\delta} \overline{\delta_M}, \quad \rho_b = \frac{\overline{\mu} \overline{\delta} \overline{\theta} \overline{\delta_M}}{\overline{\alpha} \overline{\beta}}, \quad \rho_f = \frac{\overline{\delta} \overline{\theta} (\overline{\mu} + \overline{\lambda}) \overline{\delta_M}}{\overline{\alpha} \overline{\beta}}, \\ \delta_M &= \overline{\delta_M}, \quad \alpha = \frac{\overline{\alpha} \overline{\delta_M}}{\overline{K}_{n-1}}, \quad K_i = \frac{\overline{K}_i \overline{\alpha}^i}{\overline{\theta}^i} \quad (1 \leq i \leq n-1), \\ G &= \overline{G}, \quad H = \overline{\gamma_0} - \overline{G}, \quad M = \frac{\overline{M} \overline{\delta} \overline{\theta}}{\overline{\alpha} \overline{\beta}}, \quad P = \frac{\overline{\theta} \overline{P}}{\overline{\alpha}}, \quad t = \frac{\overline{t}}{\overline{\delta_M}}\end{aligned}$$

Performing these substitutions in the raw reduced model (2), withdrawing the redundant ODE which expresses the evolution of  $H$  and removing the bars for legibility, one gets the reduced model (3):

$$\begin{aligned}\dot{G} &= \theta (\gamma_0 - G - G P^n), \\ \dot{M} &= \lambda G + \gamma_0 \mu - M, \\ \dot{P} &= \frac{n \alpha (\gamma_0 - G - G P^n) + \delta (M - P)}{\sum_{i=0}^{n-1} (i+1)^2 K_i P^i}.\end{aligned}\tag{3}$$

*Remark 1.* In order to recover [1, system (1)], it is sufficient to replace the right handside of the last equation of system (3) by its numerator.

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<sup>3</sup> Our algorithm does not handle the generic system with a symbolic  $n$ . We computed the reduced system for many different values of  $n$ , inferred the general formula and, checked afterwards that the inferred formula is correct.

Last observe that one more parameter could be removed from the above system, the software [21] shows. The above reduction is however more convenient in this paper for it permits us to directly apply the results of [1].

### 3 On the existence of Poincaré-Andronov-Hopf bifurcations

One proves in this section that no Poincaré-Andronov-Hopf bifurcation [22] arises in system (3) for meaningful<sup>4</sup> values of the parameters and the variables if and only if  $n \leq 8$ . Our proof essentially amounts to reducing the study of system (3) to that of [1, system (1)] and then applying the main result of this paper.

According to remark 1, the steady point equations of system (3) are exactly those of [1, section 4]. They write:

$$\gamma_0 = G + G P^n, \quad M = P, \quad \lambda = \frac{P - \mu G - \mu G P^n}{G}.$$

The Jacobian matrix of system (3), evaluated at the steady points<sup>5</sup> writes:

$$J = \begin{pmatrix} -\theta(1 + P^n) & -n\theta G P^{n-1} & 0 \\ \frac{P - \mu G - \mu G P^n}{G} & 0 & -1 \\ -\frac{n\alpha(1 + P^n)}{B} & -\frac{n^2\alpha G P^{n-1} + \delta}{B} & \frac{\delta}{B} \end{pmatrix}$$

where  $B = \sum_{i=0}^{n-1} (i+1)^2 K_i P^i$ .

If one clears the denominators of the last row of the Jacobian matrix  $J$ , i.e. if one lets  $B = 1$ , then one exactly gets the Jacobian matrix of [1, page 73].

*Remark 2.* The matrix  $J$  is invariant under the following transformation, where  $\ell$  denotes a nonzero arbitrary constant:

$$B \rightarrow \ell B, \quad \delta \rightarrow \ell \delta, \quad \alpha \rightarrow \ell \alpha.$$

*Remark 3.* The parameters  $K_i$  only occur in the denominator  $B$  of the Jacobian matrix  $J$ .

Assume that a Poincaré-Andronov-Hopf bifurcation occurs for system (3), for some *meaningful*<sup>6</sup> values of the system parameters and variables ( $1 \leq i \leq n-1$ ):

$$(G, P, M, \lambda, \alpha, \theta, \delta, \gamma_0, \mu, K_i) = (G^0, P^0, M^0, \lambda^0, \alpha^0, \theta^0, \delta^0, \gamma_0^0, \mu^0, K_i^0).$$

<sup>4</sup> Following [1], all variables and parameters are required to be positive apart  $\lambda$ , which may be negative but must anyway be greater than  $-\mu$ ,

<sup>5</sup> Note that the derivative of  $B$  w.r.t. to  $P$  which appears in the Jacobian of the system (3) disappears after the evaluation at the steady points since it is multiplied by a term which cancels.

<sup>6</sup> In the sense precised above.

Then, by introducing  $B^0 = \sum_{i=0}^{n-1} (i+1)^2 K_i^0 (P^0)^i$ , and according to the two remarks above, a bifurcation of [1, system (1)] occurs for the meaningful values

$$(G, P, M, \lambda, \alpha, \theta, \delta, \gamma_0, \mu, ) = (G^0, P^0, M^0, \lambda^0, \alpha^0/B^0, \theta^0, \delta^0/B^0, \gamma_0^0, \mu^0).$$

Conversely, suppose that a bifurcation of [1, system (1)] arises for some meaningful values

$$(G, P, M, \lambda, \alpha, \theta, \delta, \gamma_0, \mu) = (G^0, P^0, M^0, \lambda^0, \alpha^0, \theta^0, \delta^0, \gamma_0^0, \mu^0).$$

Using the two remarks above and taking  $\ell = 2$ , one can easily find some positive values for  $K_1^0, \dots, K_{n-1}^0$  such that  $B_0 = \sum_{i=0}^{n-1} (i+1)^2 K_i^0 (P^0)^i = 2$ . Thus a bifurcation for the system (3) occurs when

$$(G, P, M, \lambda, \alpha, \theta, \delta, \gamma_0, \mu, K_i) = (G^0, P^0, M^0, \lambda^0, 2\alpha^0, \theta, 2\delta^0, \gamma_0^0, \mu^0, K_i^0)$$

which are meaningful values. Combining the two implications, one concludes that a Poincaré-Andronov-Hopf bifurcation arises for [1, system (1)] if and only if such a bifurcation arises for the system (3), for meaningful values of the systems variables and parameters. The next proposition follows from the results of [1]:

**Proposition 1.** *For meaningful values of the parameters and the variables, no Poincaré-Andronov-Hopf bifurcation arises in system (3) if and only if  $n \leq 8$ .*

## 4 On the new quasi-steady state approximation

### 4.1 Principle of the method

Our method is based on similar ideas as in [11, 9, 12] (see [1, section 4.2] for more details). The main interest of our method is that it is fully algorithmic. One first builds a system of differential equations involving some one extra variable  $F_i$  for each fast reaction. Those extra variables are then eliminated by performing elimination. Our original implementation is based on the DIFFALG [23] package available in the standard library of MAPLE. A more recent implementation is based on the REGULARCHAINS [19] package.

This section aims at summarizing [1, section 2]. Consider the classical system of chemical reactions (4) and (5) describing the transformation of a substrate  $S$  into a product  $P$  under the action of the enzyme  $E$  (an intermediate complex  $C$  is produced):



Assume that the reaction (4) is fast. Our method consists in introducing the following system:

$$\begin{bmatrix} \dot{E} = -F_1 + k_2 C, & \dot{S} = -F_1, & \dot{C} = F_1 - k_2 C, & \dot{P} = k_2 C, & k_1 E S = k_{-1} C \end{bmatrix}.$$

Performing the elimination process (and introducing  $K = k_{-1}/k_1$ ), one gets:

$$\left[ F_1 = \frac{k_2 E S (S + K)}{K (S + E + K)}, \quad \dot{E} = \frac{k_2 E^2 S}{K (S + E + K)}, \quad \dot{P} = \frac{k_2 E S}{K}, \quad \dot{S} = -\frac{k_2 E S (S + K)}{K (S + E + K)}, \quad C = \frac{E S}{K} \right].$$

Using the conservation laws  $E + C = E_0 + C_0$  and  $S + C + P = S_0 + C_0 + P_0$  (where the subscript 0 indicates the initial concentration), assuming that  $C_0 = P_0 = 0$  and introducing  $V_m = k_2 E_0$ , further computations yield:

$$\dot{S} = -\frac{V_m S (K + S)}{K E_0 + (K + S)^2} \quad (6)$$

which differs from the Henri-Michaëlis-Menten and Briggs-Haldane formulae<sup>7</sup>:

$$\dot{S}(t) = -\frac{V_m S(t)}{K + S(t)}. \quad (7)$$

Over this easy example, the benefits are clear since the reduction is automatic and yields the formula (6) which seems more accurate than the classical reductions, especially when the condition  $S \gg E_0$  is not fulfilled. Observe that formula (7) is recovered from (6) by assuming  $S \gg E_0$ .

## 4.2 A better new reduced model

The reduced system (2) appears to be more precise than that of [1], numerical simulations show. Figure 2 shows two different numerical simulations of the variable  $G(t)$  for the same parameters and initial conditions values except for the  $k_i^-$ 's (hence for the  $K_i$ 's). In both cases, three curves are displayed: one for the initial model (1) in red, one for the raw reduced model (2) in blue and one for [1, last system page 69] in green. In both cases, the blue and the red curves almost coincide. The reduction performed in [1] is clearly less precise than the new one. As the  $K_i$ 's parameters tend towards zero, the reduction performed in [1] becomes more accurate (the rightmost picture is obtained with values of  $K_i$ 's smaller than that of the leftmost one). Thus the domain of validity of the reduction performed in [1, last system page 69] is more narrow than that of our new reduction: the  $K_i$ 's need to be small.

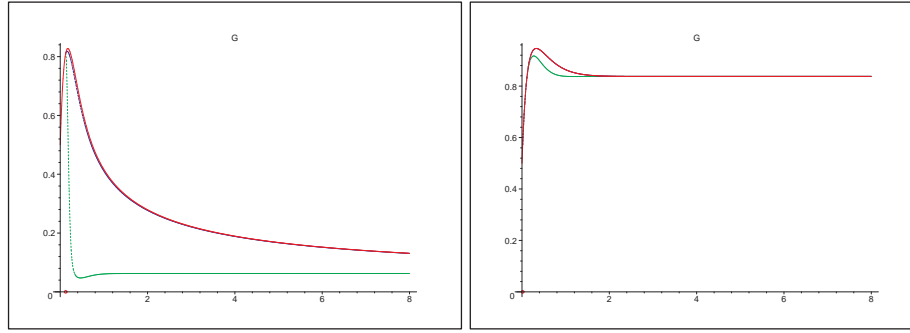
## 5 Conclusion

In this paper, the result presented in [1] is generalized by applying an algorithmic, accurate, quasi-steady state approximation method.

It is well-known that quasi-steady state approximation is useful for it permits to reduce the size of the differential system to study and the number of its parameters. But quasi-steady state approximation could also be viewed as a way to study the dynamical properties of gene regulatory networks which are

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<sup>7</sup>  $K = \frac{k_{-1}}{k_1}$  in Henri-Michaëlis-Menten's case,  $K = \frac{k_{-1} + k_2}{k_1}$  in Briggs-Haldane's.



**Fig. 2.** Numerical simulations of the variable  $G(t)$ . The horizontal axis give the time  $t$ . The vertical one gives the value of  $G$ . For both simulations,  $n = 3$ ,  $G(0) = 0.5$ ,  $M(0) = 1$ ,  $P(0) = 1$ ,  $\gamma_0 = 1$ ,  $\alpha = 1$ ,  $\theta = 10$ ,  $\rho_f = 10$ ,  $\rho_b = 5$ ,  $\beta = 50$ ,  $\delta_M = 5$ ,  $\delta_P = 10$  and  $k_i^+ = 100$  for each  $i$ . On the leftmost picture,  $k_i^- = 100$  thus  $K_i = (100/100)^i = 1$  for each  $i$ . On the rightmost one,  $k_i^- = 2000$  thus  $K_i = (100/2000)^i = 0.05^i$  for each  $i$ . The red (system 1) and the blue (system 2) curves almost coincide. The green [1, last system page 69] curve is the lower one on both pictures.

invariant for a range of reactions mechanisms since the mechanisms involved in the fast reactions may not need to be known in detail. This issue is important [24]. This shows that the development of algorithmic and accurate quasi-steady state approximation methods is an important research domain in the field of algebraic biology.

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